Putting: How a golf ball and hole interact

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Putting: How a golf ball and hole interact

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The equations of motion for a golf ball interacting with a hole on a putting surface are found. A computer is used to predict the outcome of the ball's encounter with the hole as a function of the impact parameter and the initial velocity of the ball at the rim. If the ball does not skid or bounce when it collides with the opposite rim, it is predicted that a uniform ball must travel at 1.626 m/s or less to be captured; that the British ball is easier to sink than the slightly larger American ball; that a ball with a larger moment of inertia is more difficult to sink; and that bouncing and skidding (factors that vary from green to green) result in capture at greater speeds. Experimental studies support these predictions.

I. INTRODUCTION

Putting is an important part of the game of golf; 40% to 45% of the strokes taken in professional play are putts.[1,2] In putting, the intention is to sink the ball in a hole whose diameter[3] is 4.25 in. (0.1080 m). In modern play, there are two balls in use; the American ball, with a diameter of 1.68 in. (0.0427 m), and the British ball, with a diameter of 1.62 in. (0.0411 m). (The American ball is always used in tournament play.) Golf balls are not uniform in composition, so their moments of inertia may differ from that of a uniform sphere.

In this paper, we study the encounter of a rolling golf ball with a hole. We will consider two cases: (1) a ball directed toward the center of the hole; and (2) a ball directed off-center. Both cases involve computer models of the ball–hole interaction: We have devised experimental tests of some predictions of these computer models.

In developing our models, we make some assumptions about the bouncing and skidding of the ball at the rim of the hole, assumptions that are necessary because these features of the ball's motion vary in ways that are not predictable. Therefore, the models developed here are not entirely realistic. In addition, a number of important things happen to a putted golf ball before it ever encounters the hole;[4] however, we are not presenting a complete model of putting. Nevertheless, our models show the basis for some of the interesting things that happen to putts, and the experimental tests give results consistent with the models' predictions.

We think that this study will interest those concerned with the physics of sports, especially golf, and that it also raises points that are accessible and interesting to students of elementary and intermediate mechanics.

Technical studies of putting are few. Mahoney[5] gives brief analysis of a ball interacting with a trough. Cochran and Stobbs include a chapter on putting in their admirable book.[2] So does Daish.[6] Soley's book[1] on putting statistics includes the results of some experiments on the technical aspects of putting. Resource letter[7] PS-1 lists papers on other aspects of golf.

II. BALL DIRECTED TOWARD THE CENTER OF THE HOLE

In the following discussion we refer to the American ball unless we state otherwise. In addition, we assume that air resistance and rolling friction are negligible.

A. Conditions for striking the opposite rim

A golf ball, rolling without slipping, and directed toward the center of the hole has just reached the rim of the hole. Will it remain in contact with the rim, or will it immediately begin free fall? To remain in contact with the rim, the ball's center must follow a path with radius R_b, where R_b is the ball's radius. If the initial speed is v_0, then the centripetal acceleration of the ball is v_0^2/R_b, directed downward. Gravity is the only downward force acting on the ball, so the ball loses contact with the surface if $v_0^2/R_b > g$, or

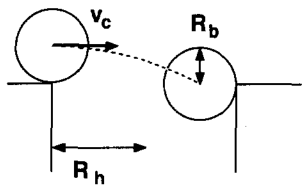

Fig. 1. The ball reaches the rim of the hole (left) with an initial speed v_c. It falls by a distance equal to its own radius R_b as it reaches the opposite rim (right). R_h is the radius of the hole.

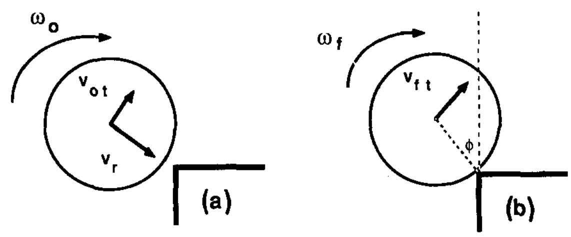

Fig. 2. Motion of ball (a) just before and (b) just after collision with the opposite rim, using the no-bounce, no-skid approximations. Figure 2(b) defines the angle ϕ; other symbols are explained in the text.

$v_0 > \sqrt{gR_b}$, where g is the acceleration due to gravity. Let us define a minimum speed:

$$v_m = \sqrt{gR_b} = 0.457 \text{ m/s}. \tag{1}$$

If the initial speed v_0 is greater than v_m, the ball loses contact with the surface when it first encounters the rim; if the initial speed is smaller than v_m, the ball rolls for a while on the rim before losing contact.

Suppose that a ball travels quickly enough at the rim to enter free fall. Will the ball reach the opposite rim before the hole captures it? Figure 1 shows a ball that reaches the front rim of the hole with a speed v_c; the ball has fallen by a distance equal to its radius R_b when it strikes the opposite rim. The time required to fall a distance R_b is $\sqrt{2R_b/g}$, and the horizontal distance traveled is $2R_h - R_b$, where R_h is the radius of the hole. So we can get the velocity v_c:

$$v_c = (2R_h - R_b)\sqrt{g/2R_b} = 1.313 \text{ m/s}. \tag{2}$$

With an initial velocity smaller than v_c, the ball will be captured before it reaches the opposite rim. However, if the ball's initial velocity is greater than v_c, the ball strikes the opposite rim of the hole, and we must consider the interaction of the ball with the opposite rim before we can decide if the ball will be captured. (For the British ball, we get $v_m = 0.449$ m/s and $v_c = 1.348$ m/s; this latter value implies that the British ball is a little easier to sink than the American ball. The main advantage of the British ball is that it has less far to fall than its more ample American cousin.)

B. Interaction with the opposite rim

When the ball collides with the opposite rim, its subsequent motion will depend on (1) how much the ball bounces, that is, on the combined coefficient of restitution of the ball and the surface; and (2) the nature of the frictional forces exerted on the ball because its rotation causes it to slide on the surface. Both the coefficient of restitution and the sliding friction are characteristics that vary from green to green. To keep our model general, we make the following approximations: (1) the ball does not bounce when it strikes the surface and (2) the impulse of the frictional forces acts on the ball so that its subsequent motion is without slippage. (The validity of these approximations is discussed in Sec. II E below.)

Figure 2(a) shows a side view of a ball approaching the opposite rim of a hole. The velocity of the ball has a radial component v_r and a tangential component v_{ot}. The ball also has an initial angular velocity ω_o. Because of the "no-bounce" approximation, the radial velocity becomes zero after the collision [see Fig. 2(b)]. At the moment of the collision, the ball is acted on by an impulse $F_t\,\Delta t$ in the tangential direction. This impulse changes the tangential momentum and the angular momentum of the ball:

$$F_t\,\Delta t = m(v_{ft} - v_{ot}),$$

and

$$F_t\,\Delta t\,R_b = I(\omega_o - \omega_f), \tag{3}$$

where m is the mass of the ball, I is the moment of inertia, and v_{ft} is the tangential velocity of the ball after the collision. The "no-skid" approximation implies that $v_{ft} = R_b\omega_f$. We combine these expressions to obtain the tangential velocity of the ball after the collision:

$$v_{ft} = (v_{ot} + \omega_o I/mR_b)/(1 + I/mR_b^2). \tag{4}$$

C. What happens after the ball strikes the opposite rim

Once the ball reaches the opposite rim [Fig. 2(b)], there are several possibilities:

(1) v_{ft} might be negative, ensuring that the subsequent motion is into the hole.

(2) v_{ft} might be positive but small, so that the ball starts to roll away from the hole. This will be the case if $v_{ft}^2/R_b < g\cos\phi$. Under these circumstances, the ball will be captured if it does not have enough mechanical energy to escape:

$$mv_{ft}^2/2 + I\omega_f{}^2/2 < mgR_b(1 - \cos\phi). \tag{5}$$

(3) v_{ft} might be positive but large enough so that the ball loses contact with the rim while moving away from the hole. This will happen if $v_{ft}^2/R_b > g\cos\phi$. Then the ball will again enter free fall, which will continue until (a) the ball escapes; or (b) the ball collides again with the rim.

D. Computer model and predictions

A flow chart organizing the alternatives described above appears in Fig. 3. Using it, we wrote a program[8] in BASIC to predict the outcome of the ball's interaction with the hole as a function of the initial speed v_o. We discuss here the predictions of this computer model.

If the ball is a uniform sphere ($I = \frac{2}{5} mR_b^2$), the program predicts that a ball will be captured if its initial speed is 1.626 m/s or less. [This speed is close to the speed 1.636 m/s obtained by replacing $2R_h - R_b$ in Eq. (1) with $2R_h$. This corresponds to assuming that the center of the ball travels ballistically to the opposite rim of the hole.[5]] The ball will escape rolling (after striking the far rim twice) for speeds up to 1.693 m/s, and it escapes flying for greater speeds. The British ball will be captured at initial speeds up to 1.653 m/s; the smaller size and smaller moment of inertia make this ball easier to sink.

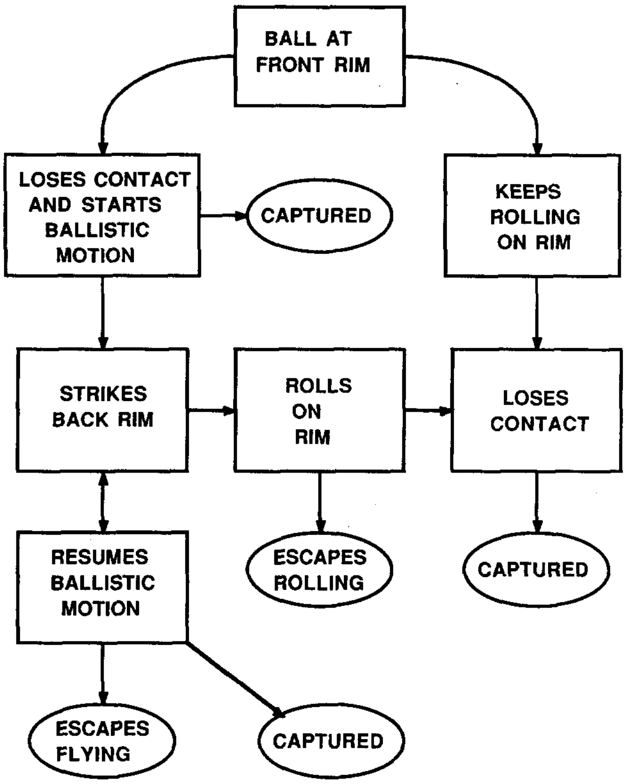

Fig. 3. Flow chart, showing the possibilities as a ball encounters with the hole head-on.

However, golf balls are not uniform spheres. Table I gives measured moments of inertia (divided by mR_b^2) for a number of different balls. The wound balls have smaller moments of inertia than the two-piece balls have.[9] (In a race down an inclined plane, a wound ball usually defeats a two-piece ball.) It is likely that other golf balls in use may have moments of inertia outside the range of those measured.

The computer model predicts that a higher moment of inertia makes a ball harder to capture. It is easy to see why this is so; the greater angular momentum of such a ball results in a bigger kick out of the hole when it strikes the opposite rim. We find that wound ball (1) with a moment of inertia of 0.375 mR_b^2, will be captured at speeds up to 1.638 m/s, whereas the two-piece ball (4), with a moment of inertia 0.415 mR_b^2, will be captured at speeds up to 1.620 m/s. That is, increasing the moment of inertia by about 10% decreases the maximum capture speed by about 1%.

Table I. Measured moments of inertia (divided by mR_b^2) for different (American) ball types. Values quoted are accurate to $\pm 0.5\%$, mostly because of imperfections in manufacture. Based on measurements by Inertia Dynamics, Inc.

Ball	I/mR_b^2
Wound(1)	0.375
Wound(2)	0.380
Two-piece(1)	0.391
Ideal sphere	0.400
Two-piece(2)	0.412
Two-piece(3)	0.413
Two-piece(4)	0.415

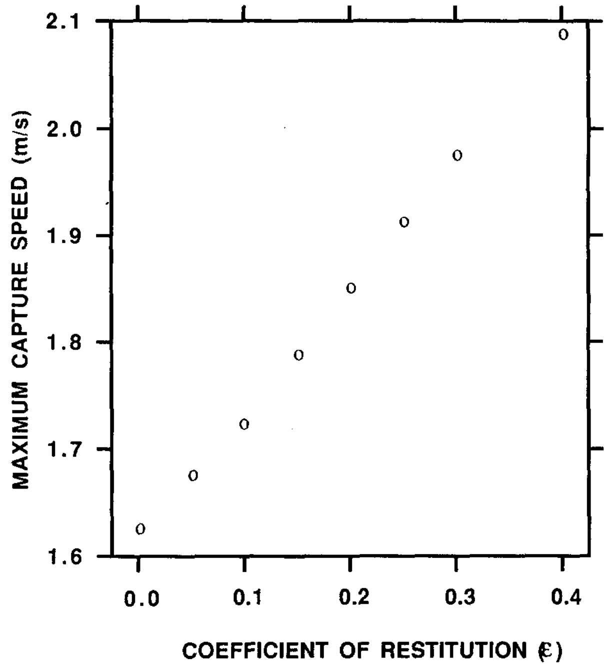

Fig. 4. How the maximum capture speed depends on the coefficient of restitution ϵ.

E. Validity of the approximations

In the previous discussion, we assumed that the ball neither bounces nor skids. We have modified the computer program described above to incorporate the coefficient of restitution of the ball colliding with the rim of the hole. (To simplify matters, we assumed that the no-skid approximation remains valid.) If the ball approaches the far rim of the hole with a velocity whose initial radial component is v_{ro}, then the radial component after the collision will be $v_{rf} = -\epsilon v_{ro}$, where ϵ is the coefficient of restitution. As the coefficient of restitution increases, the computer model predicts that the ball will be captured at greater velocities. Figure 4 shows how the maximum capture speed depends on the coefficient of restitution.

What range of values would be reasonable for the coefficient of restitution on a real green? If one drops a ball onto a green, the ball typically bounces to about 10% of its initial height. This would imply a coefficient of restitution around 0.3. However, a ball colliding with the narrow rim of a golf hole does not undergo the same kind of collision as a ball dropped onto a flat green.

How will the motion of the ball be affected if the ball skids when it strikes the opposite rim? If the ball skids during its initial impact with the opposite rim, then there will be less of a kick imparted to the ball due to its rotation; so we expect that a ball that skids should be easier to capture. (It is easy to show that a rotating ball dropped onto a flat surface will reach the same final speed no matter what the coefficient of sliding friction might be.[6] Cochran and Stobbs[2] report that when a putted ball is set in motion on a green it typically skids for the first 25% of the ball's path; Daish[6] reports similar results. However, it is difficult to say how these results relate to the cause of a ball colliding with a narrow rim.)

We have also assumed that rolling friction and air resistance are negligible; in actuality, these forces would tend to oppose the motion of the ball, making it easier to sink. However, we expect these forces are less significant while the putt interacts with the hole than the forces associated with bouncing and skidding.

III. OFF-CENTER COLLISIONS

Figure 5 shows a ball encountering a hole off-center with an impact parameter δ ($\delta = 0$ is the case treated in Sec. II). In this section, we first find the conditions for the ball to strike the opposite rim. (We did this for $\delta = 0$ in Sec. II A.) We do this because the points involved may have pedagogical interest, and because the results will help us interpret some of the predictions of the computer model. Next we describe the physical and mathematical basis of the computer model. After that, we present and discuss the results of the computer model.

A. Conditions for striking the opposite rim

A golf ball with impact parameter δ, rolling without slipping, has just reached the rim of the hole with a speed v_o. Will it remain in contact with the rim, or will it immediately begin free fall? The component of the ball's velocity toward the center of the hole is $v_o \cos \theta_o$, where θ_o is defined in Fig. 5. Consequently the centripetal acceleration of the ball (assuming it remains in contact with the rim) is $(v_o \cos \theta_o{}^2/R_b$. However, this centripetal acceleration can be no larger than g, so we can again define a minimum velocity $v_m = \sqrt{gR_b}/\cos \theta_o$, which we may write in terms of the impact parameter:

$$v_m = R_h\sqrt{gR_b/(R_h^2 - \delta^2)}. \qquad (6)$$

Figure 6 shows that the minimum velocity v_m increases as the impact parameter δ increases.

Suppose the ball travels quickly enough at the rim to enter free fall. Will the ball reach the opposite rim before the hole captures it? As in Sec. II A, the ball takes a time $\sqrt{2R_b/g}$ to fall by a distance equal to its radius R_b. Figure 7 shows the situation if the ball just makes contact with the opposite rim after falling by R_b. The horizontal distance traveled by the ball is $AB + BC = \sqrt{R_h^2 - \delta^2} + \sqrt{(R_h - R_b)^2 - \delta^2}$. Once again, we can define a velocity v_c:

$$v_c = \sqrt{g/2R_b}\left[\sqrt{R_h^2 - \delta^2} + \sqrt{(R_h - R_b)^2 - \delta^2}\right]. \qquad (7)$$

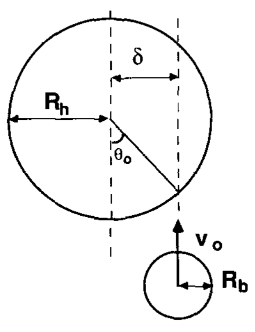

Fig. 5. The ball encounters the hole. The radius of the ball is R_b, the radius of the hole is R_h, the initial speed of the ball is v_o, the impact parameter is δ, and the initial azimuthal angle is θ_o.

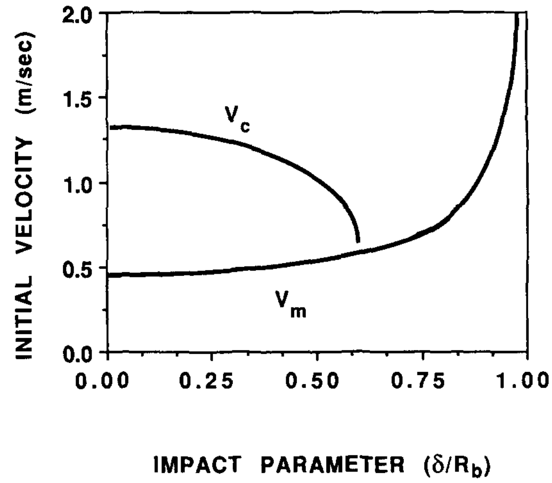

Fig. 6. The velocities v_m and v_c as functions of impact parameter δ. The ball reaches the rim with initial speed v_o. If $v_o < v_m$, the ball remains in contact with the rim. If $v_o > v_m$, the ball loses contact with the surface when it first encounters the rim. If $v_o > v_c$, the ball strikes the opposite rim before being captured.

So with an initial speed smaller than v_c, the ball will be captured before it reaches the opposite rim, and if the ball's speed is greater than v_c, the ball evidently strikes the opposite rim before being captured. Notice that we get the expected expression for v_c, Eq. (2), when $\delta = 0$. Notice too that v_c is not defined when $\delta > R_h - R_b$, which corresponds to one edge of the ball being outside the edge of the hole, making capture on the fly before striking the rim impossible. Figure 6 shows how the velocity v_c depends on the impact parameter.

B. The ball rolls on the rim

1. We set up a coordinate system

We define a coordinate system (R,θ,ϕ) as shown in Figs. 8 and 9. R is the distance between the center of the ball and the closest edge of the hole's rim; ϕ is the angle between R and the vertical, defined so that ϕ is positive when the center of the ball is inside the lip of the hole; and θ is the azimuthal angle. The associated unit vectors are $(\hat{\mathbf{R}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}})$, which form a right-handed set: $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$. These coordinates may be related to the more usual cylindrical coordinates (ρ,ϕ',z) according to

$$(\rho,\phi',z) = (R_h - R \sin \phi,\theta,R \cos \phi). \qquad (8)$$

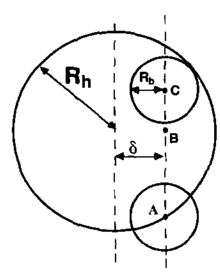

Fig. 7. The ball reaches the rim at point A. If the initial velocity of the ball is v_c, then the ball falls by a distance equal to its radius R_b as it moves horizontally at distance AB + BC.

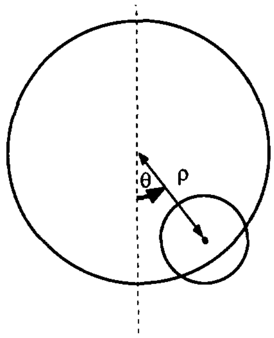

Fig. 8. Coordinates θ and ρ defined: top view of hole.

We will use ρ in the following discussion where convenient. This coordinate system is useful because the simple expression $R = R_b$ expresses the condition that the ball is in contact with the rim. There is, however, some complexity, since the unit vectors $(\hat{\mathbf{R}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}})$ are not necessarily constant in time as the ball moves.

The rotation of the ball requires three additional coordinates $(\alpha_R,\alpha_\theta,\alpha_\phi)$ denoting rotations about axes in the $(\hat{\mathbf{R}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}})$ directions. The angular velocities associated with these angles are $(\boldsymbol{\omega}_R,\boldsymbol{\omega}_\theta,\boldsymbol{\omega}_\phi)$.

2. We find the kinetic energy in terms of these coordinates

Suppose that the ball is in contact with the rim of the hole. Then its position is given by $\mathbf{R} = R\,\hat{\mathbf{R}}$. (For the moment, we neglect the constraint $R = R_b$.) To calculate the velocity, we take the derivative: So the velocity is

$$\mathbf{v} = \frac{d\mathbf{R}}{dt} = \dot{R}\hat{\mathbf{R}} + \rho\dot{\theta}\hat{\boldsymbol{\theta}} + R_b\dot{\phi}\hat{\boldsymbol{\phi}}. \tag{9}$$

Likewise, the angular velocity is

$$\boldsymbol{\omega} = \omega_R\hat{\mathbf{R}} + \omega_\theta\hat{\boldsymbol{\theta}} + \omega_\phi\hat{\boldsymbol{\phi}}. \tag{10}$$

This means that the ball's kinetic energy is

$$T = m(\dot{R}^2 + \rho^2\dot{\theta}^2 + R^2\dot{\phi}^2)/2 + I(\omega_R^2 + \omega_\theta^2 + \omega_\phi^2)/2. \tag{11}$$

3. We identify the forces acting on the ball

As the ball rolls on the rim, it is acted on by the following forces: the force of gravity mg; a normal force N directed from the surface through the center of the ball; and two friction force components f_ϕ and f_θ, which oppose slippage (the subscripts tell which way these forces point in our coordinate system). The ball is also acted on by a torque τ_R

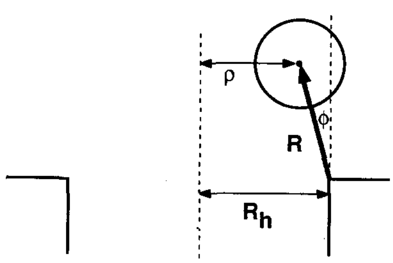

Fig. 9. Coordinates R and ϕ defined: side view of hole.

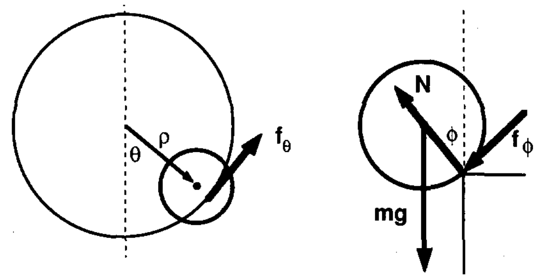

Fig. 10. The forces N, mg, f_θ, and f_ϕ acting on the ball.

which tends to oppose rotations about the R axis. The forces mg, N, f_ϕ, and f_θ are shown in Fig. 10.

4. We use Lagrange's equations

When frictional forces act on an object, Lagrange's equations of motion take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q_q, \tag{12}$$

where q is a generalized coordinate and Q_q is the associated generalized force.[10] There will be six such equations, since we have six coordinates. Considering Fig. 10, we find the generalized forces:

$$\begin{aligned} Q_R &= N - mg\cos\phi, \\ Q_\phi &= Rf_\phi + mgR\sin\phi \\ Q_\theta &= (R_h - R\sin\phi)f_\theta = \rho f_\theta \\ Q_{\alpha R} &= \tau_R, \\ Q_{\alpha\theta} &= Rf_\phi, \\ Q_{\alpha\phi} &= -Rf_\theta. \end{aligned} \tag{13}$$

If the ball rolls on the rim without slipping, then $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{R}$ and $R = R_b$. In addition, we assume that $\omega_R = 0$. So we have four constraints:

$$\begin{aligned} \omega_R &= 0, \\ \omega_\theta &= -\dot{\phi}, \\ \omega_\phi &= \rho\dot{\theta}/R_b, \end{aligned} \tag{14}$$

and

$$R = R_b.$$

Since we have six differential equations [Eqs. (12) and (13)] and four constraints [Eqs. (14)], we will end up with two independent expressions. Since R can only be R_b, the two independent expressions will give us $\ddot{\phi}$ and $\ddot{\theta}$. First, though, it is convenient to write the moment of inertia as $I = m\alpha^2R_b^2$, where α is the radius of gyration divided by the ball's radius ($\alpha^2 = \frac{2}{5}$ for an ideal sphere; in effect, Table I tabulates α^2 values for different balls). Then we get

$$\ddot{\phi} = (g\sin\phi - \rho\dot{\theta}^2\cos\phi)/[R_b(\alpha^2+1)], \tag{15}$$

$$\begin{aligned} \ddot{\theta} &= -[(\alpha^2+2)/(\alpha^2+1)]\dot{\rho}\dot{\theta}/\rho \\ &= [(\alpha^2+2)/(\alpha^2+1)]R_b\dot{\phi}\dot{\theta}\cos\phi/(R_h - R_b\sin\phi). \end{aligned} \tag{16}$$

These angular accelerations can be used to determine ϕ, θ, $\dot{\phi}$, and $\dot{\theta}$ at all times. The rotation of the ball can then be determined using Eqs. (14) above.

In addition, we obtain an expression for the normal force:

$$N = mg\cos\phi - mR_b\dot{\phi}^2 + m\rho\dot{\theta}^2\sin\phi. \quad (17)$$

As long as $N > 0$, the ball remains in contact with the rim.

C. The ball in ballistic motion

If the ball begins ballistic motion, then $N = f_\theta = f_\phi = 0$. The rotational angular momentum of the ball will be conserved. The motion of the ball will be governed by the following expressions:

$$\rho^2\dot{\theta} = L = \text{const}, \quad (18)$$

$$\ddot{R} = -g\cos\phi + R\dot{\phi}^2 - L^2\sin\phi/\rho^3, \quad (19)$$

$$R\ddot{\phi} = g\sin\phi - 2\dot{\phi}\dot{R} - L^2\cos\phi/\rho^3. \quad (20)$$

We note that angular momentum is conserved during ballistic motion. This is no surprise, but it is curious that angular momentum is not conserved while the ball rolls on the rim, since the hole's circular shape reminds us of a central potential. However, while the ball rolls on the rim, the force f_θ acts to prevent slippage, so the ball does not experience a central force.

The above equations can be used to determine R, θ, and ϕ at all times during ballistic motion. There are several possibilities to watch for during ballistic motion:

(1) The ball might fall below the rim of the hole and be captured: In terms of the coordinates, this means $\phi > \pi/2$.

(2) The ball might escape flying: In terms of the coordinates, this happens when $\phi = 0$, with $\dot{\phi} < 0$. (Notice that the ball cannot escape flying if it has just entered ballistic motion for the first time, since in that case it either hits the rim of the hole or is captured flying. However, it can escape flying after it interacts with the rim. The interaction of the ball with the rim is described below.)

(3) The ball might strike the rim: This happens when $R = R_b$ and $\dot{R} < 0$.

Equations (15) and (16) become pathological when $L = 0$. This is the case of a head-on putt. The difficulty with the equations is that θ changes abruptly from 0 to π as the ball passes the center of the hole. This case is easier to treat in normal Cartesian coordinates.

D. How the ball collides with the rim

Once the ball ends its ballistic motion by colliding with the rim, we have to determine the ensuing motion. We apply a no-bounce, no-skid approximation. We eliminate the bounce by setting $\dot{R} = 0$ when the ball collides. We eliminate the skid by imagining that the ball receives an impulse during the collision so that its motion immediately afterward obeys the condition $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{R}$.

Just before the collision with the rim, the ball has the following angular velocities: $(\dot{\theta}_1, \dot{\phi}_1, \omega_{\theta 1}, \omega_{\phi 1})$. (Actually, the ball will in general also have the angular velocity ω_{R1}, but we will assume this does not affect the collision.) The tangential velocity of the ball (that is, the component of its velocity perpendicular to $\hat{\mathbf{R}}$) is $\mathbf{v}_{\tan 1} = R_b\dot{\phi}_1\hat{\boldsymbol{\phi}} + \rho\dot{\theta}_1\hat{\boldsymbol{\theta}}$. After the collision, the new angular velocities and tangential velocity will be $(\dot{\theta}_2, \dot{\phi}_2, \omega_{\theta 2}, \omega_{\phi 2})$ and $\mathbf{v}_{\tan 2} = R_b\dot{\phi}_2\hat{\boldsymbol{\phi}} + \rho\dot{\theta}_2\hat{\boldsymbol{\theta}}$.

At the instant of collision, we assume that the ball is acted on by an impulse $(F_\theta\Delta t)\hat{\boldsymbol{\theta}} + (F_\phi\Delta t)\hat{\boldsymbol{\phi}}$. This impulse changes the tangential momentum of the ball:

$$m\rho(\dot{\theta}_2 - \dot{\theta}_1) = (F_\theta\Delta t),$$

$$mR_b(\dot{\phi}_2 - \dot{\phi}_1) = (F_\phi\Delta t). \quad (21)$$

Likewise, the impulse changes the angular momentum of the ball:

$$I(\omega_{\theta 2} - \omega_{\theta 1}) = R_b(F_\phi\Delta t), \quad (22)$$

$$I(\omega_{\phi 2} - \omega_{\phi 1}) = -R_b(F_\theta\Delta t). \quad (23)$$

Eliminating the unknown impulse between these equations and applying Eqs. (14), the no-skid condition, we get the following conditions after the collision:

$$\dot{\theta}_2 = (\dot{\theta}_1 + \alpha^2 R_b\omega_{\phi 1}/\rho)/(1+\alpha^2), \quad (24)$$

$$\dot{\phi}_2 = (\dot{\phi}_1 - \alpha^2 R_b\omega_{\theta 1}/\rho)/(1+\alpha^2). \quad (25)$$

By the way, the previous discussion bypasses a tricky point. The ball has undergone ballistic motion. When it left the rim to enter ballistic motion, it had an angular velocity with $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\theta}}$ components. Later, just an instant before it struck the rim after ballistic motion, it has the same angular velocity. However, because it has moved to a new position, our coordinate system has moved around so that the angular velocity does not have the same $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\theta}}$ components at the new position. This matter is tedious, but since it is simply an issue of describing a vector in a coordinate system that has rotated, we will not discuss it here.

E. What happens to the ball

When the first ball encounters the rim of the hole, it will either lose contact with the rim and enter ballistic motion, or it will continue to roll on the rim. If the ball is rolling on the rim, it will continue to do so until it escapes rolling or is captured rolling, or until it loses contact with the rim. If the ball is in ballistic motion, it will continue until it is captured by the hole or until it strikes the rim again. When it strikes the rim, the no-bounce, no-skid approximation is invoked. Having struck the rim, the ball will either continue rolling on the rim or lose contact with the rim. If it loses contact a second time, it enters ballistic motion, which will be terminated either by capture, escape flying, or striking the rim again.

It is possible to predict the velocity and path of a ball after it has escaped the hole, but we will not discuss this here.

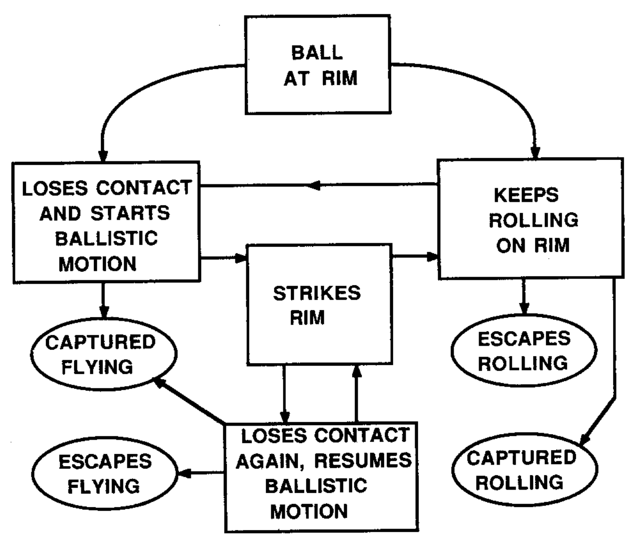

Fig. 11. Flow chart, showing the possibilities as the ball encounters the hole off-center.

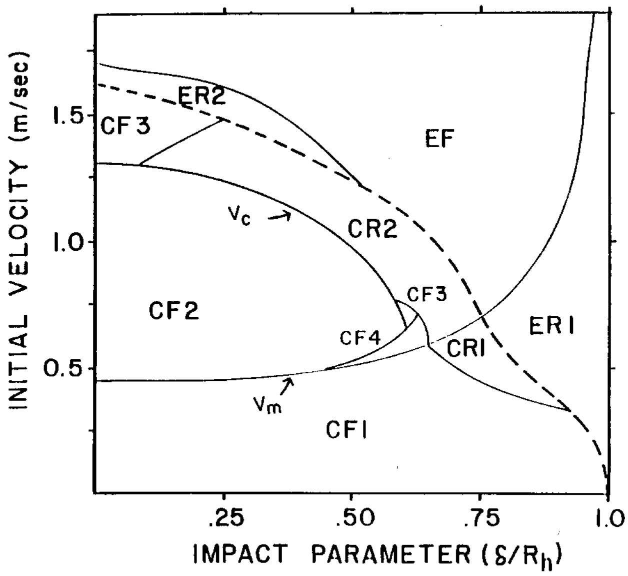

Fig. 12. Putting outcomes as a function of initial speed and the dimensionless impact parameter δ/R_h, based on the computer program. Balls below the dashed line are captured; those above escape. CF = captured flying; CR = captured rolling; EF = escaped flying; ER = escaped rolling. Arabic numerals and the speeds v_m and v_c are explained in the text: See also Fig. 6.

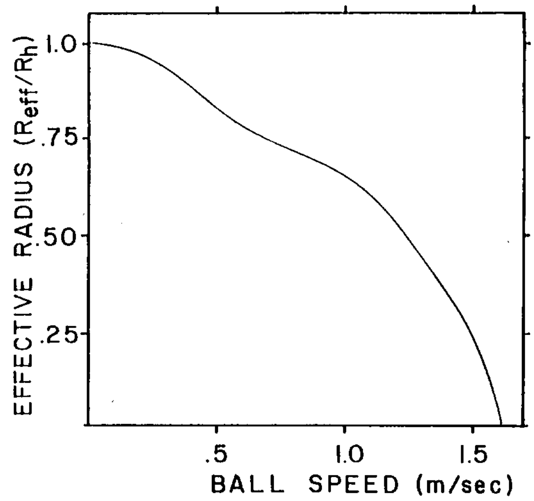

Fig. 13. The effective radius of the hole as a function of initial ball speed.

F. Computer model and predictions

A flow chart for the computer model is given in Fig. 11. The computer program[8] was written in FORTRAN.

Figure 12 gives a "phase diagram" of the computer predictions of the possible outcomes as a function of the initial velocity v_o and the impact parameter δ. In this figure, CF means that the ball is captured flying; CR means that it is captured rolling; ER means that it escapes rolling; and EF means that it escapes flying. *The most important feature of this figure is the dashed line: All putts below this line are captured, and all putts above it escape.* We see that capture is less likely as the impact parameter δ increases. Two other features to notice in this figure are the lines labeled v_m and v_c. These velocities, discussed in detail in Secs. II A and III A above (see Fig. 6), are verified by the computer model: Balls with $v > v_m$ lose contact with the front rim as soon as they reach it, and balls with $v < v_c$ are captured before they reach the opposite rim. Thus CF1 represents balls that roll on the front rim before they are captured flying; CF2 represents balls that lose contact with the surface as soon as they reach the front rim and that are captured before striking the opposite rim of the hole; CF3 represents balls that are captured flying after striking the opposite rim; and CF4 represents balls that lose contact with the surface when they first encounter the front rim and that are then captured after striking the front rim a second time. CR1 represents balls that are captured rolling and that never lose contact with the rim; CR2 denotes balls that are captured rolling after undergoing ballistic motion. ER1 denotes balls that escape rolling without losing contact with the rim; ER2 shows those which escape rolling after striking the opposite rim.

The interaction of the ball with the hole is analogous to certain interactions in nuclear physics; just as one may consider the capture cross section for subatomic particle impinging on a nucleus, so too may one consider the cross section for capture of a golf ball by a hole. However, the nuclear cross section is the effective area of a nucleus, whereas a golf hole should be discussed in terms of its effective radius, since an approaching ball is constrained to move on the surface of the green.[11] In these terms, one could say that for a ball with a velocity over 1.626 m/s, the effective radius of the hole is zero, since the ball is not captured.[12] Figure 13 gives the effective radius of the hole as a function of the ball's initial velocity v_o. As expected, the hole "looks" smaller as the ball's velocity increases; a ball with a velocity of 1.23 m/s encounters a hole whose effective radius is half its actual radius. A reader who has followed the arguments this far will have little trouble discerning which regions of Fig. 12 correspond to elastic and inelastic scattering of the ball from the hole.

This computer model, like the model for a head-on putt, predicts that the American ball is slightly harder to sink than the British ball, and that a ball with a large moment of inertia is slightly harder to sink than one with a smaller moment of inertia.

IV. TESTING THE COMPUTER PREDICTIONS

We constructed an artificial golf hole to test the computer results. A hole of the appropriate diameter was drilled in a plywood board. The board was faced with a corklike material 0.5 cm thick. This material reduced, but did not entirely eliminate, the bouncing of the ball as it rolled on the surface. Golf balls rolled down an inclined plane, onto the surface, and through a pair of photogates (Pasco model 9206) before interacting with the hole. The photogates allowed measurement of the ball's speed. We measured the impact parameter by observing how the rolling balls marred a narrow ridge of powder (manufactured by Johnson and Johnson) in front of the hole. The balls did not always follow the same path, even when released under supposedly identical conditions; this may be due to the dimpling of the balls.

This artificial hole has a coefficient of restitution around 0.3. For collisions with the rim, however, the value may be smaller. (Care was taken to support the plywood firmly, to prevent the board from recoiling and reducing the bounce.)

Figure 14 presents results of the experimental test. The

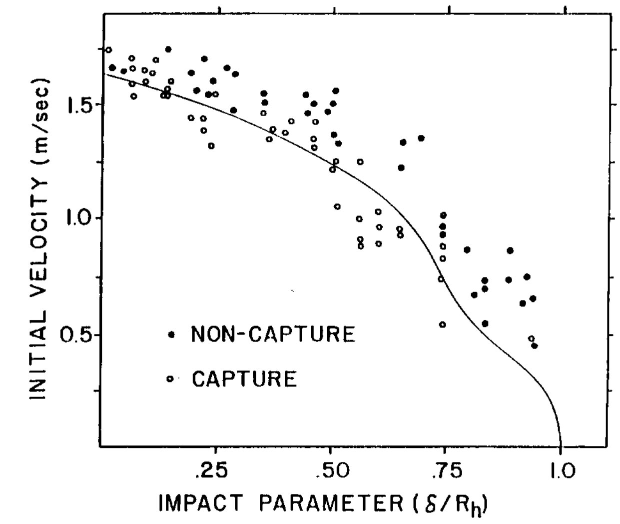

Fig. 14. Experimental test of the computer model. Ball capture (○) and noncapture (●) as a function of the initial ball speed and impact parameter δ. Balls below the solid line should sink, according to the computer prediction; those above should escape.

solid line represents the computer prediction: Balls below this line should sink, and those above should escape. We note that some balls sink although their initial speeds and impact parameters put them above this line. This is probably due to bouncing and skidding of the balls at the rim of the hole. In addition, the balls may be slowing as they approach the hole so that the actual speed at the rim will be slightly less than the speed the photogates measure.

We also extensively studied head-on putts ($\delta = 0$) for the balls with the smallest and largest moments of inertia given in Table I. These experiments support the qualitative prediction that an increase in the moment of inertia makes the ball harder to sink (assuming the speed remains the same). We have not tested whether the British ball is easier to sink than the American ball.

ACKNOWLEDGMENTS

I would like to thank Clyne Soley for lending his Stimpmeter and offering a copy of his book; Peter Zidnack for lending a wealth of published material on the technical aspects of putting; Al Einarsson for constructing the artificial golf hole; J. F. Mahoney for sending an extended copy of his paper; Jeffrey Callen, John Hildebrand, Fred Holmstrom, Frank Kearly, Frank Thomas (of the USGA), Allen Tucker, and the Northern California Section of the AAPT for useful suggestions and comments; Neil Brock and Anthony Sarto for experiments on golf balls; Will Lockhart for experimental studies of the ball–hole interaction; and Mark Shiring of Inertia Dynamics for measuring of moments of inertia of golf balls.

I wish to dedicate this paper to Walter Ogier on the occasion of his retirement from the Physics Department of Pomona College: may $\delta = 0$ for all his putts!

[1] C. Soley, *How Well Should You Putt?* (Soley Golf Bureau, San Jose, CA, 1977).

[2] A. Cochran and J. Stobbs, *The Search for the Perfect Swing* (Lippincott, New York, 1968).

[3] English units are closely associated with golf.

[4] B. Holmes, "Dialogue concerning the Stimpmeter," Phys. Teach. **24**(7), 401–404 (1986); J. Witters and D. Duymelinck, "Rolling and sliding resistive forces on balls moving on a flat surface," Am. J. Phys. **54**, 80–83 (1986).

[5] J. F. Mahoney, "Theoretical analysis of aggressive golf putts," Res. Q. Exercise Sport **53**, 165–72 (1983); J. F. Mahoney, "The measurement of error in aggressive golf putts" (unpublished).

[6] C. B. Daish, *The Physics of Ball Games* (English Universities Press Ltd., London, 1972).

[7] C. Frohlich, "Resource letter PS-1: Physics of sports," Am. J. Phys. **54**, 590–593 (1986).

[8] Listings of computer programs used in this paper are available from the author on request.

[9] Golf magazines occasionally list golf balls according to their construction. See, for example, J. B. Mattes, "Best ball," Golf **30**, 92–94 (1988).

[10] K. R. Symon, *Mechanics* (Addision-Wesley, Reading, MA, 1971), 3rd ed., pp. 365–366.

[11] G. Hageseth and F. McCormack, "The poorman's macroscopic scattering analyzer," Am. J. Phys. **37**, 204–210 (1969); and C. Anderson and H. von Baeyer, "Theory of a ball rolling on a $1/\rho$ surface of revolution," Am. J. Phys. **38**, 140–145 (1970).

[12] This corresponds to an energy of 3×10^8 GeV.

$F = ma$: PHYSICAL LAW OR DEFINITION OF FORCE?

At this point I paused to try to work out the Newtonian mechanics of the process, the role of the mudcap's mass in the delivery of energy to the boulder, filling a page with little diagrams and equations from freshman physics at MIT, then later trying to remember whether $F = ma$ was merely a definition of F or whether it actually told something useful.

Richard L. Meehan, *Getting Sued and Other Tales of the Engineering Life* (MIT, Cambridge, 1981), pp. 162–163.